Online-Appendix zu

„Portfolio Optimization and Ambiguity Aversion“

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Appendix

Appendix A: Derivation of the optimal portfolio weights

A.1 Mean-variance portfolio

For the mean-variance portfolio (Markowitz, 1952), the optimal portfolio weights w_{MV} can be calculated as follows:

$$w_{MV} = \frac{1}{\gamma}\Sigma^{-1}\hat{\mu} \qquad (A.1.1)$$

in which the vector of expected excess returns is considered. This computation is applied for the model incorporating parameter and model uncertainty. For the model, only taking parameter uncertainty into account, a risk-free asset doesn't exist and the optimal portfolio weights are derived as follows:

$$w_{MV} = \frac{1}{\gamma}\Sigma^{-1}(\hat{\mu} - \hat{\mu}^0\mathbf{1}_N). \qquad (A.1.2)$$

In the absence of the risk-free asset, the expected returns on the zero-beta portfolio are subtracted from the expected returns. The zero-beta portfolio can be written as:

$$\hat{\mu}^0 = \frac{B-\gamma}{A} \qquad (A.1.3)$$

with A = $\mathbf{1}_N^T\Sigma^{-1}\mathbf{1}_N$ and B = $\hat{\mu}^T\Sigma^{-1}\mathbf{1}_N$.

A.2 Minimum-variance portfolio

For the minimum-variance portfolio, the optimal portfolio weights w_{MIN} can be calculated as follows:

$$w_{MIN} = \frac{1}{A}\Sigma^{-1}\mathbf{1}_N. \qquad (A.2.1)$$

A.3 Bayes-Stein portfolio

The optimal portfolio weights of the Bayes-Stein portfolio w_{BS} are a weighted average of the portfolio weights of the mean-variance portfolio and the minimum-variance portfolio and can be calculated as follows:

$$w_{BS} = \phi_{BS}w_{MIN} + (1-\phi_{BS})w_{MV} \qquad (A.3.1)$$

in which ϕ_{BS} is the shrinkage factor, which can be written as:

$$\phi_{BS} = \frac{\nu_\mu}{n + \nu_\mu} \qquad (A.3.2)$$

and ν_μ is the precision of the prior on μ and can be written as:

$$\nu_\mu = \frac{N + 2}{(\hat{\mu} - \mu_{MIN})^T \Sigma^{-1} (\hat{\mu} - \mu_{MIN})} \qquad (A.3.3)$$

in which μ_{MIN} is the mean of the minimum-variance portfolio (Jorion, 1986).

A.4 Ambiguity-averse approach with parameter uncertainty only

The optimal portfolio weights for GUW's ambiguity-averse approach w_{AA}, incorporating parameter uncertainty only, can be written as:

$$w_{AA} = \frac{1}{\gamma} \Sigma^{-1} \left(\frac{1}{1 + \frac{\sqrt{\varepsilon}}{\gamma \sigma_P^*}} \right) \left[\hat{\mu} - \frac{B - \gamma \left(1 + \frac{\sqrt{\varepsilon}}{\gamma \sigma_P^*} \right)}{A} \mathbf{1}_N \right] \qquad (A.4.1)$$

in which ε is defined as

$$\varepsilon = \epsilon \frac{(n-1)N}{n(n-N)}. \qquad (A.4.2)$$

n is the length of the in-sample period (window length) and σ_P^* is the positive real solution to the following polynomial equation:

$$A\gamma^2 \sigma_P^4 + 2A\gamma\sqrt{\varepsilon}\sigma_P^3 + (A\varepsilon - AC + B^2 - \gamma^2)\sigma_P^2 - 2\gamma\sqrt{\varepsilon}\sigma_P - \varepsilon = 0 \qquad (A.4.3)$$

with $A = \mathbf{1}_N^T \Sigma^{-1} \mathbf{1}_N$, B = $\hat{\mu}^T \Sigma^{-1} \mathbf{1}_N$ and C = $\hat{\mu}^T \Sigma^{-1} \hat{\mu}$.

(A.4.1) can then be written as (13), in which the optimal portfolio weights w_{AA} are a weighted average of the portfolio weights of the mean-variance portfolio and the minimum-variance portfolio. The shrinkage factor $\phi_{AA}(\epsilon)$ can be computed as follows:

$$\phi_{AA}(\epsilon) = \frac{\sqrt{\varepsilon}}{\gamma \sigma_P^* + \sqrt{\varepsilon}}. \qquad (A.4.4)$$

A.5 Ambiguity-averse approach with parameter and model uncertainty

For GUW's ambiguity-averse approach, incorporating both, parameter and model uncertainty, the optimal portfolio weights for the risky assets w_a and for the factor portfolios w_b are given by the following system of equations:

$$w_a = max\left[1 - \frac{\sqrt{\epsilon_a}}{\sqrt{g(w_b)^T \Sigma_{aa}^{-1} g(w_b)}}, 0\right] \frac{1}{\gamma} \Sigma_{aa}^{-1} g(w_b), \quad \text{(A.5.1)}$$

$$w_b = max\left[1 - \frac{\sqrt{\epsilon_b}}{\sqrt{h(w_a)^T \Sigma_{bb}^{-1} h(w_a)}}, 0\right] \frac{1}{\gamma} \Sigma_{bb}^{-1} h(w_a), \quad \text{(A.5.2)}$$

with

$$g(w_b) = \hat{\mu}_a - \gamma \Sigma_{ab} w_b, \quad (A.5.3)$$

$$h(w_a) = \hat{\mu}_b - \gamma \Sigma_{ba} w_a. \quad (A.5.4)$$

$\hat{\mu} = \begin{pmatrix} \hat{\mu}_a \\ \hat{\mu}_b \end{pmatrix}$ presents the expected excess returns and $\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$ presents the variance-covariance matrix of the risky assets and the factors.

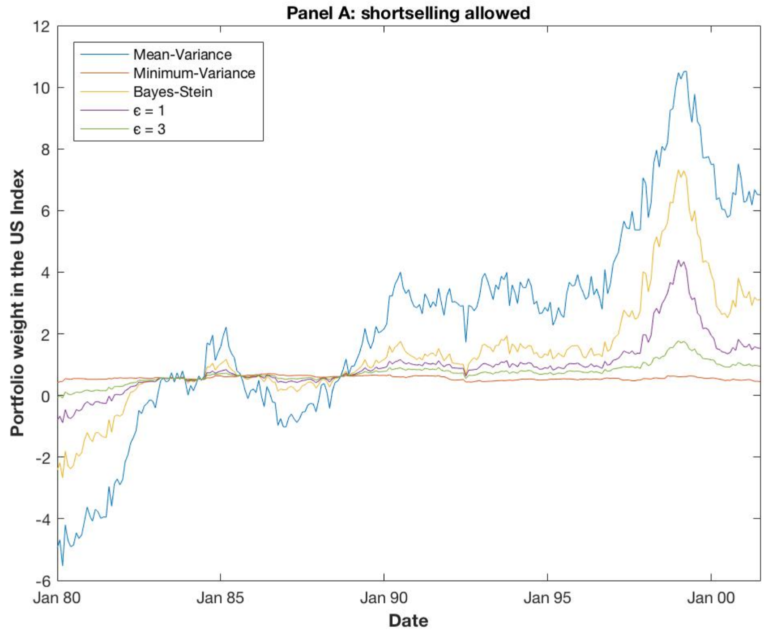
Appendix B: Replication of the results of GUW – Portfolio weights

A graphical representation of the turnover ratios for the different portfolio optimization strategies is presented in Figure B.1, for both cases, when short sales are allowed (Panel A) and when short sales are not allowed (Panel B). In both cases the portfolio weights in the US index over time is shown.

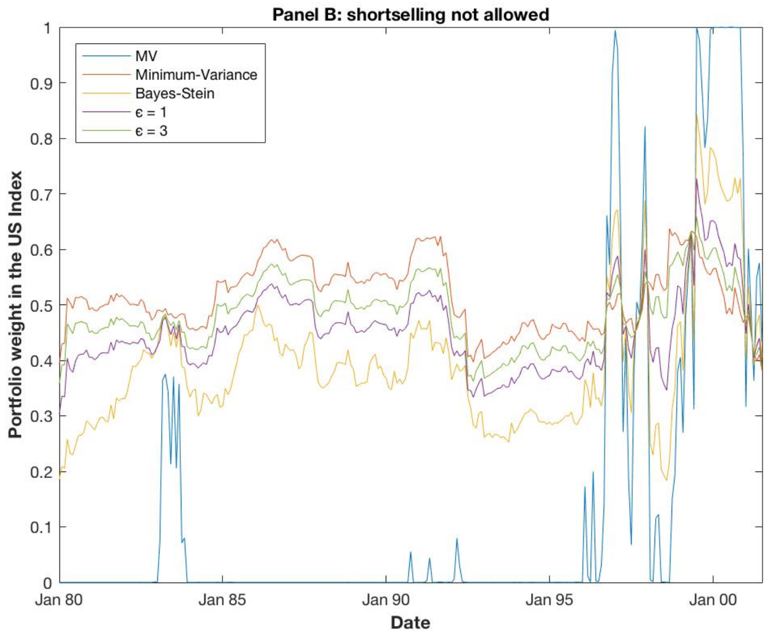
Figure B.1: Portfolio weights in the US index over time

This figure presents the portfolio weights in the US index over time for different portfolio optimization strategies. The portfolio strategies considered are the mean-variance strategy (blue line), the minimum-variance strategy (red line), the Bayes-Stein strategy (yellow line), the ambiguity-averse approach with $\epsilon = 1$ (purple line) and the ambiguity-averse approach with $\epsilon = 3$ (green line).

Panel A: short sales allowed



Panel B: short sales not allowed



The graphical representation confirms the results presented in Chapter 5.2.1.

Appendix C: DAX30 – parameter and model uncertainty

Table C.1 reports the out-of-sample performance of the mean-variance strategy, the minimum-variance strategy, the Bayes-Stein strategy, the ambiguity-averse approach and the 1/N rule for a portfolio optimization problem, in which the investment universe consists of 16 DAX30 stocks and where both, parameter and model uncertainty are considered.

Table C.1: Parameter and model uncertainty – DAX30 stocks

This table reports the out-of-sample Sharpe ratio for the returns on different portfolio optimization strategies. The parameter ϵ_a presents the ambiguity about the asset returns and varies between 0 and 3 and ϵ_b presents the ambiguity about the returns of the factor portfolio also varying between 0 and 3. The parameter ω presents the investor's belief in the validity of the CAPM, where ω = 0 means the investor doesn't believe in the CAPM and ω = 1 means the investor believes dogmatically in the CAPM. The values for the Sharpe ratios are expressed as percentage per month. The timespan considered extends from January 1973 to December 2016.

Application to the DAX30 stocks							
Strategy	**Sharpe ratio**						
Mean-Variance	0.0321						
Minimum-Variance	0.1377						
Bayes-Stein	0.0466						
Naive Diversification	0.0689						
Ambguity-averse	**With ω = 0**			ϵ_b			
Sharpe ratio	**0.00**	**0.50**	**1.00**	**1.50**	**2.00**	**2.50**	**3.00**
ϵ_a = 0.00	0.0531	-0.0158	-0.0158	-0.0158	-0.0158	-0.0158	-0.0158
ϵ_a = 0.25	0.1116	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 0.50	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 0.75	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 1.00	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 1.50	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 2.00	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 2.50	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 3.00	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	With ω = 1						
ϵ_a = 0.00	0.1117	0.0487	0.0487	0.0487	0.0487	0.0487	0.0487
ϵ_a = 0.25	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 0.50	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 0.75	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 1.00	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 1.50	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 2.00	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 2.50	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ϵ_a = 3.00	0.1117	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000