Online-Appendix zu

„Backtesting the Expected Shortfall"

Konstantin Spring

Universität Konstanz

Junior Management Science 6(3) (2021) 590-636

Appendices

A Robustness of critical values of the Test 2

As outlined in subsection 4.2, Acerbi and Szekely (2014) argues that critical values of the test statistic Z of their proposed Test 2 are roughly stable across different estimated return loss variables $\widehat{L}_t$. Thus, they conclude that a bootstrap procedure to determine the test decision is redundant. Furthermore, based on simulated critical values within their article, which are depicted in Table 2 within this thesis, they propose critical values of the test statistic of $Z^* = 0.7$ and $Z^* = 1.8$ for backtesting significance levels of $\kappa = 0.05$ and $\kappa = 0.0001$.

In order to complement the analysis by Acerbi and Szekely (2014) on the stability of critical values, I use log-return losses of the S&P 500 index[32] to forecast ES values for a 250 day window between the 03.10.2018 and the 01.10.2019. Moreover, I consider two different estimation models. For the estimation of $\widehat{ES}_{t,0.975}$, I use the previous 250 log-returns to fit both a normal distribution as well as a t-distribution in *Python* and calculate the ES values with respect to the fitted distributions. For both estimation models, I simulate the distribution of the test statistic under the null hypothesis according to the procedure outlined in subsection 4.2 using $M = 10000$ simulation trials.

Figure 12 depicts the bootstrap distributions of the test statistics for both estimated models. The left sub-plot shows the simulated distribution of the test statistic using normally distributed return loss estimates, while the right sub-plot shows the respective distribution with t-distributed return loss estimates.

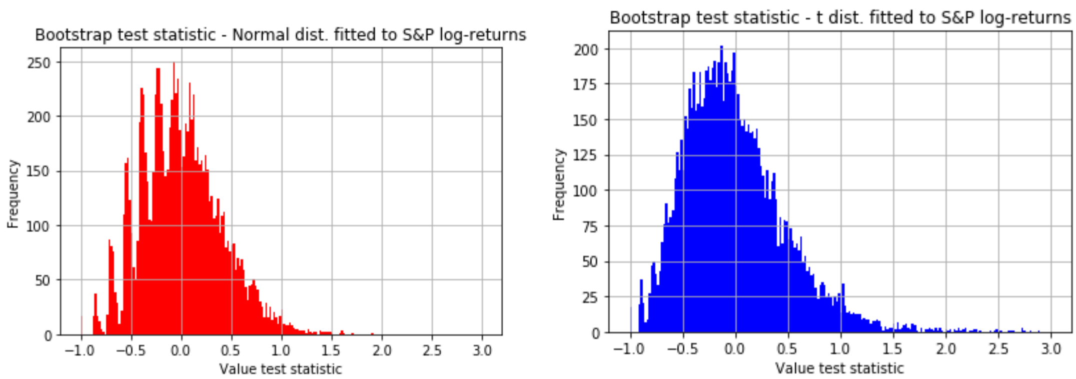

Figure 12: Simulated distribution of test statistic Z for two different risk estimation models fitted to log-returns of the S&P 500 between 03.10.2018 and 01.10.2019. On the upper sub-plot return losses $\widehat{L}_t$ are fitted to a normal distribution on the lower sub-plot to a t-distribution, respectively.

At a first glance, indeed both distributions appear to be fairly comparable with most probability mass at test statistic values between -1 and 1. Nevertheless, it can be ob-

[32]Given that P_t is the price of the S&P 500 at time t, the log return loss l_t is given by $l_t = -(\ln(P_t) - \ln(P_{t-1}))$. See more on the calculation of log-return losses of the S&P 500 in chapter 6 of this thesis.

served that the test statistic related to the t-distributed return loss estimates $\widehat{L}_t$, exhibits more data points at the far right tail of the distribution. Indeed, even ten out of the 10000 values of Z are not depicted in the right sub-plot as they exceed a value of three. The maximum value of the test statistic for the t-distribution estimation model is given at a value of 12.45, whereas the maximum for the normal estimation model is given at a far lower value of 1.91. Correspondingly, it does not seem to be surprising, that also the simulated critical values, which correspond to the quantiles of the respective test statistic distribution, differ for both estimation models. For values of $\kappa = 0.05$ and $\kappa = 0.0001$, the critical values, using the normal distribution for estimation, are at values of $Z^* = 0.71$ and $Z^* = 1.70$ respectively, which is roughly in line with the values suggested by Acerbi and Szekely (2014). Nevertheless, for the model using t-distributed return losses the corresponding simulated critical values are given by $Z^* = 0.88$ and $Z^* = 5.60$. Especially, the value at a significance level of $\kappa = 0.0001$ heavily diverges from that suggested by Acerbi and Szekely (2014). Indeed, for the considered period of S&P 500 log return losses, fitting a t-distribution for every estimate $\widehat{L}_t$ leads to an average value of $\nu = 2.54$ degrees of freedom. Although, Acerbi and Szekely (2014) argues that t-distributed return loss variables with $\nu = 3$ or even less degrees of freedom display rather extreme scenarios, they still seem to be relevant for practical scenarios, given the obtained simulation results.

Overall, it appears that especially for rather extreme quantiles of the test statistic distribution under the null hypothesis critical values are not sufficiently stable over different estimation models.

B Variable description

This section summarizes the most important variables and notation used throughout this thesis. All relevant variables together with a short description and a reference to their first appearance within this thesis are depicted in Table 17 below. Note that Table 17 only contains variables which are defined in a more general sense and used in different settings within this thesis. Thus, for example not all variables are included, which are only relevant for a certain backtesting methodology.

Variable	**Description**	**Reference within the thesis**
L, $F(y)$, $f(y)$ for $y \in \mathbb{R}$	Unconditional return loss variable and the related CDF and PDF function	see Notation 2.3
L_t. $F_t(y)$, $f_t(y)$ for $y \in \mathbb{R}$	Conditional return loss variable at time t, based on information up to time $t-1$ and the related conditional CDF and conditional PDF function.	see Notation 2.3
$q_\alpha(L_t \mid \mathscr{F}_{t-1})$	α-quantile of conditional return loss variable L_t	see Notation 2.3
ρ	Risk measure	see Definition 2.1
$\mathscr{A}_\rho$	Acceptance set related to risk measure ρ	see Definition 2.2
α	Confidence level related to VaR or ES risk measure	see Definition 2.4
$VaR_{t,\alpha}$	VaR of conditional return loss variable L_t at confidence level α	see Definition 2.4
$ES_{t,\alpha}$	ES of conditional return loss variable L_t at confidence level α	see Definition 2.5
ψ	Admissible risk spectrum	see Definition 2.6
$\mathscr{M}_{t,\psi}$	Spectral risk measure of conditional return loss variable L_t based on admissible risk spectrum ψ	see Definition 2.7
$\phi(y)$, $\Phi(y)$ for $y \in \mathbb{R}$	PDF and CDF function of standard normal distribution	see for example formula (2.6)
$\phi^{\text{skewed}}(y)$ for $y \in \mathbb{R}$	PDF function of skewed normal distribution	see formula (5.3)
ν	Degrees of freedom of t-distributed variable	see chapter 5.3
$g_\nu(y)$, $t_\nu(y)$ for $y \in \mathbb{R}$	PDF and CDF function of standard t-distribution with ν degrees of freedom	see formulas (6.8) and (6.9)

Table 17: – Table continuous on next page –

Variable	**Description**	**Reference within the thesis**
S	Strictly consistent scoring function (context of elicitability)	see Definition 2.13
T	Backtesting horizon	see Notation/ Assumptions 3.1
μ_t, σ_t	Mean and volatility of conditional return loss variable L_t	see chapter 5.2
$l_t \sim L_t$	Realized return loss distributed according to L_t In chapter 6, l_t corresponds to a log return loss	see Notation/ Assumptions 3.1
$\widehat{L}_t$	Forecast of conditional return loss variable L_t based on information up to time $t-1$	see Notation/ Assumptions 3.1
$\widehat{\rho}_t$, $\widehat{VaR}_{t,\alpha}$, $\widehat{ES}_{t,\alpha}$, $\ldots$	Risk measure forecast of conditional return loss variable L_t based on information up to time $t-1$	see Notation/ Assumptions 3.1
$(a_t^1, \ldots a_t^k)$	Set of k auxiliary variables at time t required as additional input for a backtest	see Notation/ Assumptions 3.1
κ	Backtesting significance level	see for example chapter 3,2
$I_t(\alpha)$	Violation indicator of the VaR at time t at confidence level α	see Definition 3.4
X_ψ^T	Spectral risk measure violation rate over backtesting horizon T related to risk measure $\mathscr{M}_\psi$	see Definition 4.3
r_t	Exceedance residual at time t	see formula (4.45)
M	Number of simulation trials for bootstrap decision	see chapters 4.2, 4.3 and 4.5
MC	Number of simulation trials in size/power simulation	see chapter 5
λ	Decay factor in RiskMetrics approach (chapter 6) Shape parameter of skewed normal distribution (chapter 5)	see formula (6.11) or formula (5.3)
$S_{FHS,t}$	Sample required for ES estimation in FHS estimation approach	see formula (6.12)

Table 17: Description of the most important variables together with a reference within the thesis.