Online-Appendix zu

„ Analysis of Green Bonds"

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liquidity premium. In this case, the option is executed for $LP = GP$. As the liquidity premium cannot change over time, the option is executed as soon as liquidity effects and the green premium cancel each other out. The resulting maximum value of the switch option before execution, $ST^{\max}$, is equal to zero in this scenario. For $LP < GP$, there is no chance that the option is executed as it implies a certain negative yield differential Δy. Therefore, the value of the option is equal to zero in this case as well.

Table 6: Option value at execution for different σ

σ	LP	$ST^{\max}$
0	8	0
0.002	10.53	2.53
0.004	13.28	5.28
0.008	18.81	10.81
0.010	21.26	13.26

The table shows the values of $ST^{\max}$ for different σ based on $GP = 8$bp, a risk-free rate of $r_f = 200$bp, $a = 11.9$, $T = 3.1$ years and a trinomial tree length of $N = 791$.

Finally, Table 7 shows $ST^{\max}$ for different times to maturity T. In the model, this increases the length of the trinomial tree because $\Delta t = \frac{T}{N} = \frac{1}{250}$ is held constant. The results indicate a lower maximum value of the switch option $ST^{\max}$ for longer maturities T. This is explained by the decreasing likelihood of the stochastic liquidity premium realizing an outcome lower than LP. Therefore, the switch option is executed for a lower expected liquidity premium LP reducing its maximum value $ST^{\max}$.

Table 7: Option value at execution for different T

T	LP	$ST^{\max}$
1	12.35	4.35
5	12.06	4.06
10	11.51	3.51
20	10.76	2.76
30	10.31	2.31

The table shows the values of $ST^{\max}$ for different T (constant Δt) based on $GP = 8$bp, a risk-free rate of $r_f = 200$bp, $\sigma = 0.0031$ and $a = 11.9$. Changes in T affect the tree length N, as Δt is hold constant with $\Delta t = \frac{T}{N} = \frac{1}{250}$.

5.3 Sensitivity Analysis of Bond Yields

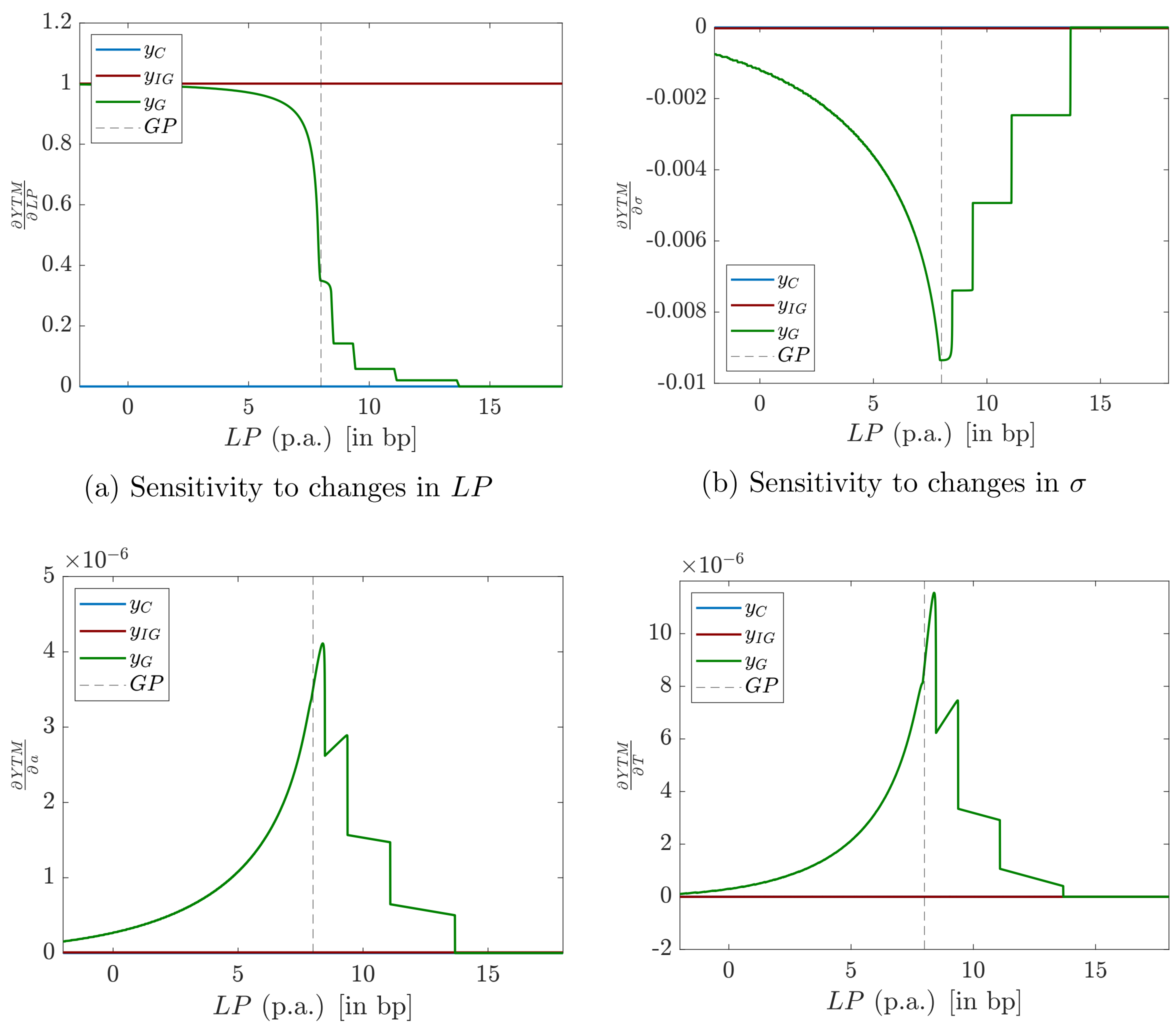
The sensitivities of the initial yield to maturity to changes in the model parameters are estimated using finite differences that is motivated by a Taylor approximation. This approximation is required because a closed-form solution is not available due to the non-closed form of the model. Following **?**, a symmetric approximation of the first partial derivative of the yield y_0 with regard to the model parameters is computed, as this approach yields a lower order truncation error compared to forward or backward approximation. In its general form, the first derivative can be estimated using,

$$\frac{\partial y_0(x)}{\partial x} \approx \frac{y_0(x+h) - y_0(x-h)}{2h}, \tag{25}$$

where h denotes a small and constant value and x the parameter of interest, while the other model parameters are hold constant. The resulting sensitivities are displayed in Figure 10. The figures indicate that the sensitivity of the German green bond G has a continuous part, and a discontinuous part with jumps when LP assumes values above a certain threshold. The number of observed jumps in the figures for G coincide with $j_{max} = 4$ (or $-j_{min}$) of the calibrated model. One viable explanation might be that nodes in the tree switch to the value of the conventional bond, if the liquidity premium assumes a high enough value so that $P^G < P^C$ (see Equation 15). This also explains the continuous part on the left-hand side of the figures, as a switch scenario does not occur for low values of LP.

Figure 10a describes how much units the yield changes, if LP changes by one unit. The yield of the illiquid green bond y^{IG} changes by one basis point, if LP increases by one basis point, while y^C is unaffected by changes in LP. The sensitivity of y^G ranges between 1 and 0. This aligns with the notion that the German green bond is valued as a conventional bond if LP is sufficiently high and valued as a counterfactual bond without switch option, if LP is sufficiently low, assuming a constant GP. In those cases, the stochastic process for LP either cannot assume values where y^G is lower than y^C, or where the switch option is executed. Figure 10b implies that a higher instantaneous volatility σ decreases y^G. This is because the downside potential is restricted by the switch option, while a lower realized liquidity premium reduces y^G. The parameter a describes the mean reversion rate of the stochastic process. Therefore, this sensitivity is inversely related with the sensitivity of y^G to σ. Finally, an increase in T, increases the yield y^G as well. Based on the absolute size of the sensitivities, the evaluation suggests that changes in σ and LP have the strongest impact on the model results. In light of the evaluation, it should be noted that the sensitivities only reflect the impact of small changes in the parameters. Further, their changes and thus the effect on the model results is restricted by their plausible range. Nevertheless, the model outcome might be significantly larger or smaller, if different estimates for those parameters are chosen.

Figure 10: Model sensitivities



(a) Sensitivity to changes in LP

(b) Sensitivity to changes in σ

(c) Sensitivity to changes in a

(d) Sensitivity to changes in T

The model results displayed in the figures above are based on a green premium of $GP = 8$bp, a risk-free rate of $r_f = 200$bp, $\sigma = 0.0031$, $a = 11.9$, $T = 3.1$ years, a trinomial tree length of $N = 791$ and $h = 0.00001$

5.4 Limitations

The above discussed model for the green bond yields provides a first insight into the potential effects of the switch option between green and conventional bonds, which was pioneered by the German twin bond approach. However, the model is subject to some limitations that are discussed in the following.

First, the model cannot decompose observed green bond yields $\hat{y}_G$ into the different components suggested by the model. Namely, the observed yield of the respective conventional twin $\hat{y}_C$, the liquidity premium LP, the green premium GP and the added-value of the switch option ST. This means that a calibration of the model parameters is not straightforward and proxies need to be applied instead. Moreover, this impedes the validation of the model results based on actual observations.

Another possible limitation can be the assumed process for the liquidity premium and its translation into a trinomial tree representation. For example, the Vasicek process in Equation 2 assumes a constant volatility and is, in addition to a mean-reversion parameter, defined by its first two moments. This means that it cannot accommodate possible volatility clusters or skewness that is introduced by jumps in the liquidity premium, as shown in Figure 6. Moreover, deriving the trinomial tree representation, we assume a maximum range from LP_{jmin} to $LP_{j_{max}}$ for the liquidity premium to ensure positive tree probabilities. This creates an upper and lower threshold that the liquidity premium cannot exceed. However, increasing the volatility of the process may provide a first idea of the possible implications when accounting for these effects, as it increases the overall dispersion of the stochastic premium.

Finally, the model assumes a constant risk-free rate r and green premium GP. While adding additional complexity to the model by introducing more flexible (e.g., stochastic or time-dependent) components might improve the calibration to observed yield spreads, this is not relevant for the main objective of this dissertation to better understand the potential impact of the switch option.

6 Conclusion

The goal of this dissertation is to provide a theoretical model for the pricing of green bonds that are based on the German twin bond approach. The focus here is on improving the understanding of the potential effects of introducing a switch mechanism between green bonds and their conventional counterparts. For this purpose, a non-closed form solution was derived that decomposes the yield differential into three effects: A liquidity premium, a green premium and the added value of the switch option. The model assumes a stochastic liquidity premium that follows a Vasicek process in discrete time, a constant green premium as well as a constant risk-free rate. The switch mechanism is modelled by assuming the theoretical value of conventional bonds as a lower limit for the green bond prices. For the model calibration the term structures of German Bundesanleihen and Pfandbriefen are used to obtain a proxy for the stochastic liquidity premium.

The main learning from the model is that the switch option can in certain conditions increase the value of the green bonds, which corresponds to a lower yield. Based on the calibration of the model, a maximum added-value of 4.1 bp before the execution of the option was identified. This translates to a maximum value of about 2 mn. EUR assuming a green bond with a 5 bn. EUR issuance volume. This means that issuers adopting the twin bond concept may be able to secure lower costs of capital compared to a traditional green bond concept that does not provide the switch option. For investors the concept reduces their exposure to potential liquidity risks by using the liquid conventional bonds to create a lower limit for the green bond price. The model improves the understanding of the twin bond concept and thereby fills a gap in the literature. From a practical perspective, the model implications may assist issuers in the design choice of their green bond framework. For example, Denmark decided to adopt the twin bond concept, including a switch mechanism, which supports the potential benefits of this approach.

Green bonds are one important instrument to finance the transition to a more sustainable economy. In light of the significant growth of the green bond market in recent history and the competing frameworks, it is crucial to elaborate on their respective advantages and disadvantages. While this work contributes to the understanding of the twin bond switch mechanism, the current model can be further developed. On the one hand, an improved proxy for the liquidity premium and a larger sample of historic data may affect the calibration results, which can impact the size of the evaluated effects. On the other hand, a more sophisticated stochastic process for the liquidity premium and less restrictive assumptions in its discrete representation may increase the precision of the model results. In a broader context, one should evaluate if a high issuance volume of green bonds can affect the liquidity of similar conventional bonds, and whether a potential effect vanishes for lower volumes. If such effects are found, this would support the relevance of the twin bond approach with switch option to mitigate liquidity risks, as lower overall issuance volumes may be required. Otherwise, ensuring a critical volume that is high enough to avoid liquidity costs may be a viable alternative to this concept.

A Appendix

A.1 Derivation of Trinomial Tree Model

We derive the probabilities for the trinomial tree following the assumptions in **?**. For the default branching method (i.e., $j_{min} < j < j_{max}$), the condition for the expected change notates as,

$$\begin{aligned} p_u \cdot \Delta s + p_m \cdot 0 + p_d \cdot (-\Delta s) &= \mathbb{E}[dLP_t] \\ &= -a \cdot j \cdot \Delta s \cdot \Delta t \end{aligned} \tag{26}$$

Dividing by Δs and solving for p_d yields

$$p_d = p_u + a \cdot j \cdot \Delta t. \tag{27}$$

We use this result in the condition for the variance

$$\begin{aligned} p_u \cdot \Delta s^2 + p_m \cdot 0^2 + p_d \cdot \Delta s^2 &= \sigma^2 \cdot \Delta t + a^2 \cdot j^2 \cdot \Delta s^2 \cdot \Delta t^2 \\ p_u \cdot \Delta s^2 + (p_u + a \cdot j \cdot \Delta t) \cdot \Delta s^2 &= \sigma^2 \cdot \Delta t + a^2 \cdot j^2 \cdot \Delta s^2 \cdot \Delta t^2 \end{aligned} \tag{28}$$

and solve for p_u to obtain

$$p_u = \frac{1}{2}\sigma^2 \cdot \frac{\Delta t}{\Delta s^2} + \frac{1}{2}a^2 \cdot j^2 \cdot \Delta t^2 - \frac{1}{2}a \cdot j \cdot \Delta t. \tag{29}$$

We use $\Delta s = \sigma\sqrt{3\Delta t}$ or $\Delta s^2 = \sigma^2 \cdot 3\Delta t$ to obtain

$$\begin{aligned} p_u &= \frac{1}{2}\sigma^2 \cdot \frac{\Delta t}{\sigma^2 \cdot 3\Delta t} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 - a \cdot j \cdot \Delta t\right) \\ &= \frac{1}{6} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 - a \cdot j \cdot \Delta t\right). \end{aligned} \tag{30}$$

We use this result to obtain the probability p_d as

$$\begin{aligned} p_d &= p_u + a \cdot j \cdot \Delta t \\ &= \frac{1}{6} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 + a \cdot j \cdot \Delta t\right). \end{aligned} \tag{31}$$

Finally, we use $p_u + p_m + p_d = 1$ to obtain

$$\begin{aligned} p_m &= 1 - p_u - p_d \\ &= \frac{2}{3} - a^2 \cdot j^2 \cdot \Delta t^2. \end{aligned} \tag{32}$$

At the limits of the trinomial tree (i.e., j_{min} and j_{max}), the branching structure changes as displayed in Figure 5.

First, we compute the probabilities for the lower limit j_{min}. For this, we change the condition for the expected change to,

$$\begin{aligned} p_u \cdot 2\Delta s + p_m \cdot \Delta s + p_d \cdot 0 &= \mathbb{E}[dLP_t] \\ &= -a \cdot j \cdot \Delta s \cdot \Delta t, \end{aligned} \tag{33}$$

as the liquidity premium cannot decrease any further. We obtain

$$p_m = -2p_u - a \cdot j \cdot \Delta t, \tag{34}$$

which we substitute into the new condition for the variance

$$\begin{aligned} p_u \cdot 4\Delta s^2 + p_m \cdot \Delta s^2 + p_d \cdot 0^2 &= \sigma^2 \cdot \Delta t + a^2 \cdot j^2 \cdot \Delta s^2 \cdot \Delta t^2 \\ p_u \cdot 4\Delta s^2 + (-2p_u - a \cdot j \cdot \Delta t) \cdot \Delta s^2 + p_d \cdot 0^2 &= \sigma^2 \cdot \Delta t + a^2 \cdot j^2 \cdot \Delta s^2 \cdot \Delta t^2. \end{aligned} \tag{35}$$

We solve the equation for p_u and use $\Delta s^2 = \sigma^2 \cdot 3\Delta t$ to obtain

$$\begin{aligned} p_u &= \frac{1}{2}\sigma^2 \cdot \frac{\Delta t}{\sigma^2 \cdot 3\Delta t} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 + a \cdot j \cdot \Delta t\right) \\ &= \frac{1}{6} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 + a \cdot j \cdot \Delta t\right). \end{aligned} \tag{36}$$

We use this result to obtain the probability p_m as

$$\begin{aligned} p_m &= -2p_u - a \cdot j \cdot \Delta t \\ &= -\frac{1}{3} - a^2 \cdot j^2 \cdot \Delta t^2 - a \cdot j \cdot \Delta t - a \cdot j \cdot \Delta t \\ &= -\frac{1}{3} - a^2 \cdot j^2 \cdot \Delta t^2 - 2a \cdot j \cdot \Delta t. \end{aligned} \tag{37}$$

Finally, we use $p_u + p_m + p_d = 1$ to obtain

$$\begin{aligned} p_d &= 1 - p_u - p_m \\ &= \frac{7}{6} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 + 3a \cdot j \cdot \Delta t\right). \end{aligned} \tag{38}$$

In the same fashion, we can compute the probabilities for j_{max}. In this case, we change the condition for the expected change to,

$$p_u \cdot 0 + p_m \cdot (-\Delta s) + p_d \cdot (-2\Delta s) = \mathbb{E}[dLP_t] \tag{39}$$

and the condition for the variance to

$$p_u \cdot 0 + p_m \cdot \Delta s^2 + p_d \cdot 4\Delta s^2 = \sigma^2 \cdot \Delta t + a^2 \cdot j^2 \cdot \Delta s^2 \cdot \Delta t^2. \tag{40}$$

In this case, we obtain

$$\begin{aligned} p_u &= \frac{7}{6} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 - 3a \cdot j \cdot \Delta t\right) \\ p_m &= -\frac{1}{3} - a^2 \cdot j^2 \cdot \Delta t^2 + 2a \cdot j \cdot \Delta t \\ p_d &= \frac{1}{6} + \frac{1}{2}\left(a^2 \cdot j^2 \cdot \Delta t^2 - a \cdot j \cdot \Delta t\right) \end{aligned} \tag{41}$$

Based on the calibration result of the model in section 4.3, we use these formulas to compute the probabilities for the trinomial tree. The probabilities are displayed in Table 8.

Table 8: Trinomial tree probabilities

	j	p_u	p_m	p_d
j_{max}	4	0.9008	0.0093	0.0900
	3	0.1058	0.6465	0.2477
	2	0.1238	0.6577	0.2184
	1	0.1441	0.6644	0.1914
	0	0.1667	0.6667	0.1667
	-1	0.1914	0.6644	0.1441
	-2	0.2184	0.6577	0.1238
	-3	0.2477	0.6465	0.1058
j_{min}	-4	0.0900	0.0093	0.9008

The table shows the probabilities for the trinominal tree using $\sigma = 0.0031$, $a = 11.9$, $T = 3.1$ years, $N = 791$ and assuming $\Delta s = \sigma\sqrt{3\Delta t}$.

A.2 Additional Robustness Tests for Switch Option Value

Table 9: Option value at execution for different interest rate r

r	LP	$ST^{\max}$
-100	12.06	4.06
0	12.06	4.06
100	12.06	4.06
200	12.06	4.06

The table shows the values of $ST^{\max}$ for different r based on $GP = 8$bp, $\sigma = 0.0031$, $a = 11.9$, $T = 3.1$ years and a trinomial tree length of $N = 791$.

Table 10: Option value at execution for different rounding precision

Precision	LP	$ST^{\max}$
4	8	0.35
5	10.31	2.36
6	12.06	4.06
7	12.35	4.35
8	12.35	4.35

The table shows the values of $ST^{\max}$ for different rounding precisions, measured in digits after the decimal point. A precision of six digits equals 0.01bp. The further parameters are $GP = 8$bp, a risk-free rate of $r_f = 200$bp, $\sigma = 0.0031$, $a = 11.9$, $T = 3.1$ years and a trinomial tree length of $N = 791$.

A.3 Additional Estimation Results and Statistics

Table 11: Summary statistics for BAS of German twin bonds

	Mean	SD	Min	p25	Median	p75	Max	N
BAS_{G2050}	0.67	0.31	0.00	0.50	0.60	0.90	1.50	131
BAS_{C2050}	0.57	0.27	0.10	0.40	0.50	0.90	1.00	131
BAS_{G2031}	0.57	0.33	0.20	0.30	0.50	0.90	1.30	47
BAS_{C2031}	0.29	0.11	0.20	0.20	0.30	0.30	0.80	47
BAS_{G2030}	0.70	0.36	0.20	0.40	0.60	0.90	1.70	305
BAS_{C2030}	0.29	0.24	0.10	0.20	0.20	0.30	1.70	305
BAS_{G2025}	1.27	0.85	0.20	0.60	1.15	1.40	3.30	262
BAS_{B2025}	0.57	0.26	0.30	0.40	0.50	0.70	2.40	262

The table shows the summary statistics (i.e., mean, standard deveation (SD), minimum, 25th percentile (p25), median, 75th percentile (p75), maximum and number of observations (N)) for the bid-ask spread of the closing yields in basis points of the German twin bonds displayed in Table 1. The data is retrieved from Refinitiv Eikon (Accessed: 10.11.2021) and covers the period from 09.09.2020 to 10.11.2021.

Table 12: Summary statistics for yield of German twin bonds

	Mean	SD	Min	p25	Median	p75	Max	N
y_{G2050}	15.96	13.95	-9.80	4.70	17.00	26.70	41.40	127
y_{C2050}	6.12	17.73	-48.60	-8.00	4.30	20.60	44.80	566
y_{G2031}	-25.34	8.40	-40.70	-35.10	-23.70	-17.90	-13.50	43
y_{C2031}	-29.71	12.59	-49.80	-41.90	-30.10	-18.80	-9.70	103
y_{G2030}	-47.15	12.86	-66.90	-59.10	-48.50	-35.40	-21.20	301
y_{C2030}	-43.02	12.69	-64.10	-54.30	-45.50	-31.80	-15.70	358
y_{G2025}	-72.22	6.80	-86.70	-76.80	-72.20	-67.40	-54.50	258
y_{C2025}	-69.38	7.02	-83.70	-74.55	-69.50	-64.65	-46.60	344

The table shows the summary statistics for the yields in basis points of the German twin bonds displayed in Table 1. The data is retrieved from Refinitiv Eikon (Accessed: 04.11.2021).

Table 13: Summary statistics for Δy between German twin bonds

	Mean	SD	Min	p25	Median	p75	Max	N
Δy_{2050}	-3.95	0.45	-5.70	-4.30	-4.00	-3.60	-3.10	126
Δy_{2031}	-4.41	0.42	-5.50	-4.70	-4.40	-4.20	-3.20	43
Δy_{2030}	-4.76	1.64	-7.60	-6.30	-5.10	-3.30	-1.40	301
Δy_{2025}	-3.36	1.88	-8.50	-4.40	-3.10	-2.30	0.10	258

The table shows the summary statistics for the yield spread (i.e., $\Delta y = y_G - y_C$) in basis points between the German twin bonds displayed in Table 1. The data is retrieved from Refinitiv Eikon (Accessed: 04.11.2021).

Table 14: Summary statistics for liquidity proxy

	Mean	SD	Min	p25	Median	p75	Max	N
LP_{2050}	93.08	9.61	50.54	89.01	93.66	98.55	116.80	549
LP_{2031}	48.76	6.47	36.26	43.77	47.51	52.58	80.15	549
LP_{2030}	48.61	6.31	36.21	43.95	47.55	51.92	78.25	549
LP_{2025}	48.66	6.44	38.01	44.23	47.49	50.66	72.24	549

The table shows the summary statistics for the estimated proxy of the liquidity premium in basis points. The data is based on published yield curves by the **?** and covers the period from 02.09.2019 until 01.11.2021.

Table 15: Summary of OLS estimation results for Vasicek process

	2050	2031	2030	2025
LP_{t-1}	0.7662***	0.9023***	0.9078***	0.9534***
	(0.0537)	(0.0231)	(0.0223)	(0.0159)
Constant	0.0022***	$0.0005 * **$	0.0004***	0.0002***
	(0.0005)	(0.0001)	(0.0001)	(0.0001)
R-squared	0.5863	0.8150	0.8250	0.9089
N	548	548	548	548

Robust standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$

The table shows the estimation results for the process of the liquidity premium specified in Equation 22. The dependent variable is the daily measured liquidity procxy LP_t and the independent variables are a constant and the lagged value LP_{t-1}. The data is based on published yield curves by the **?** and covers the period from 02.09.2019 until 01.11.2021.

A.4 List of German Twin Bonds

Table 16: German Twin Federal Securities

Name	First Issue Date	Last Issuance	Maturity Date	Coupon	Outstanding	Type	ISIN	RIC
2021 (2050) Bund/g	18.5.2021	11.05.2021	15.8.2050	0.00%	6.0 bn. EUR	Green	DE0001030724	DE103072=
2019 (2050) Bund	23.8.2019	18.08.2021	15.8.2050	0.00%	29.0 bn. EUR	Brown	DE0001102481	DE110248=
2021 (2031) Bund/g	10.9.2021	20.10.2021	15.8.2031	0.00%	6.5 bn. EUR	Green	DE0001030732	DE103073=
2021 (2031) II Bund	18.6.2021	10.11.2021	15.8.2031	0.00%	26.5 bn. EUR	Brown	DE0001102564	DE110256=
2020 (2030) Bund/g	9.9.2020	02.09.2020	15.8.2030	0.00%	6.5 bn. EUR	Green	DE0001030708	DE103070=
2020 (2030) II Bund	19.6.2020	18.11.2020	15.8.2030	0.00%	30.5 bn. EUR	Brown	DE0001102507	DE110250=
Bobl/g	6.11.2020	04.11.2020	10.10.2025	0.00%	5.0 bn. EUR	Green	DE0001030716	DE103071=
Bobl	10.7.2020	02.12.2020	10.10.2025	0.00%	25.0 bn. EUR	Brown	DE0001141828	DE114182=

Source: **?** and Refinitiv Eikon (Accessed: 21.11.2021)